Approximation of Fixed Points of Contactive Operators by a Stochastic Iterative Algorithm

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***ABSTARCT*:**
We prove a strong convergence result using stochastic iterative scheme with contractive operators associated with error terms. The iterative process is faster, simpler which improves other algorithms, and for the class of mapping in a more global sense.

KEYWORDS: *Contractive operator, stochastic algorithm, fixed point, strong convergence.*

1. INTRODUCTION:

Let $\mathfrak{H}$ be a real Hilbert space with the usual inner product $\langle\cdot,\cdot\rangle$ which induces the norm$\|\cdot\|$. Let $\emptyset\{C\}$ be a non empty, closed and convex subset of $\mathfrak{H}$. Let T be a non linear mapping of $\emptyset$, the set of fixed points of $\tau(T)$ is defined as $Fix(\tau) = \{Z \in \emptyset : \tau_z = Z\}$ (1)

In the sense of Brower-Petryshyn, $\tau: \emptyset \to \emptyset$ is α-strict pseudo contractive if $\exists\ 0 \le \alpha < 1$ such that $\|\tau_x - \tau_y\|^2 \le \|x - y\|^2 + \alpha\|(k-\tau)_x - (k-\tau)_y\|^2 \quad , \forall\, x, y \in \emptyset$ (2)

2 PRELIMINARIES:

Our purpose in this work is to prove a convergence result for approximating fixed points of contractive operator by stochastic approximation, using iterative process defined in (2). Our approach is different from that of Mann(see[2]). In this way, our results improve and generalize corresponding results of (4 and 5).

The problem of finding fixed point of non expansive mappings by Manns algorithm (see[1]) has been investigated in copious literatures (see[2]).

The Mann iterative is defined by sequence:

$x_1 \in \emptyset\,,\ x_{n+1} = \gamma_n x_n + (1-\gamma_n)\tau x_n\,,\ \forall\, n \ge 1\ \ and\ \ (\gamma_n)_{n\in N} \in (0,1)$ (3)

Note that if $\tau: \emptyset \to \emptyset$ is a nonexpansive mapping with a fixed point in a closed and convex subset of a uniformly convex, with a Frechét differentiable norm , then the sequence (x_n) generated by Mann's algorithm converges weakly to a fixed point of τ globally (see[2]).

On the other hand, iterative algorithms for α- strict pseudo-contractions are still developing compare to that of nonexpansive mappings. Strict pseudo-contractions have many applications due to its direct links with inverse strongly monotone operators. If A is a strongly monotone operator, then $\tau = I - A$is a strict pseudo-contraction as such the problem can be reduced to locating zeros for A in a fixed point problem for τ (see[3] and [4]). Safer Hussain Khan in [5] and [6] proved a convergence result for approximating fixed point of the class of contractive-likeoperators, which improve the generalize result.

Definition1: A mapping $T: G \to H$ is said to weakly contractive of $\psi\ Class -\ C_{\phi(t)}$ on a closed convex subset $G \subset \mathcal{H}$ if there exists a continuous and increasing function $\emptyset(t)$ such that
$\forall\ x, y \in G, \|Tx - Ty\| \leq \delta\|x - y\| + \lambda(\|T_x - y\|)$

Definition 2: A mapping $T: G \to H$ is said to be non expansive on the closed convex subset $G \subset \mathcal{H}$ if for all $x, y \in G\ then\ \|Tx - Ty\| \leq \|x - y\|$. The class of weakly contractive maps is contained in the class of nonexpansive maps.

In this work, we formulate approximation of fixed points of contractive operator by a stochastic iterative process. In Imoru and Olatinwo (see[6]), gave a general definition: An operator T is called contractive-like operator if there exists a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\lambda: [0, \infty) \to [0, \infty)$, with $\lambda(0) = 0$ such that for each$x, y \in$ G, $\|T_x - T_y\| \leq \delta\|x - y\| + \lambda(\|T_x - x\|)$. (4)

We introduced a stochastic iteration process by setting

$$\|T_x - T_y\| \leq \delta\|x - y\|\ \ with\ T_x = x\ \ at\ \ x = x_0$$

such that $\|T_x - T_y\| \leq \delta\|x - y\| + \lambda(\|x_0 = 0\|)$, which guarantee the existence of unique fixed point. With some assumptions, we have iteration process as $\|T_x - T_y\| \leq \delta\|x - y\|\ \ and\ y = 0, such\ that\ \|T_x\| \leq \delta x,\ where\ \delta = \sum_{i=1}^{n} a_{ij} \leq 1$ (5)

We investigate the stochastic approximation of the form :

$$x^{k+1} = x^k - a_x \nabla f(x^k) \quad (6)$$

the factor $a_x \nabla f(x^k)$ is an infinite-dimensional vector of random observations of the clearance operator at random points $x_n \in G\ \forall k \geq 1$ on the same probability space $(\Omega, \mathrm{A}, \mathbb{P})$.

3. FORMULATION OF STOCHASTIC APPLICATION ALGORITHM

$Let\ d^k = \xi_k + \eta_k,\ \ where\ \xi_k = x_n - Tx_n,\ and\ \eta_k$ is a sequence of independent random vectors with the conditions:
$E\|u\| < \infty\ and\ E < a, u >=< a, E_u >\ \forall a \in H$ (see[11])
Let E be the expectation operator. If $E\|u\| < \infty$, then
$E\langle a, u\rangle = \langle a, Eu\rangle\ \forall\ a \in H$ Given any linear continuous monotone map $T: H \to H$, there exists a continuous, convex and everywhere Frechet differentiable scalar function
$f: H \to R$, such that $f(x) = \frac{1}{2}\langle Tx, x\rangle - \langle b. x\rangle$ (7)
The gradient mapping ∇f, uniquely determined for a given $T\ and\ b$ coincides with $Tx^* - b\ \forall x$. Finding x^* such that
$Tx^* = b\ and\ \langle Tx^* - b, y - x^*\rangle \geq 0\ \forall y \in K$ [7], is equivalent to finding the unique zero $x^*\ of\ \nabla f(x^*)$ such that $0 = \nabla f(x^*)$ (8)
Given that ∇f is a monotone operator defined on Hilbert space into itself then the iterative scheme $x^{k+1} = x^k - a_k \nabla f(x^k)$ (9)
converges strongly to a zero of ∇f for a suitable $\{a_k\}$.
We consider a stochastic approximation algorithm. The usual form of stochastic approximation algorithm is a minimum point $x^* \in H$, of a function $f: H \to R$ is of the form $x^{k+1} = x^k - a_k d^k$ (10)

Where $\nabla f(x^k) = \int d(\theta) P_{x_k}(dt)\ and\ P_{x_k}$ is the usual probability law of $d(x^k) = d^k,\ d^k$ is approximation of the gradient $\nabla f(x^k)$, and $\{a_k\}$ is a sequence of positive scalars that decreases to zero (see [9], [10] and [12]). The important part of (10) is the gradient approximationd^k. We applied objective function measurements and requires the minimization of a continuous and convex function. Given a sequence of random vectors d^k, computed from different data points: $P_j \in H$, for each $k\ and\ j = 1, \dots, N$ such that d^k approximates $\nabla f(x^k)$ strongly and in mean square in the sense that the estimated vectors are minimized as follows:

Definition 3: The sequence of random vectors d^k , $\nabla f(x^k)$ is strongly approximates if $E\|d^k - \nabla f(x^k)\| = 0\ \ for\ each\ k$ (11)
and consistent with $\nabla f(x^k)$ in mean square if:
$E\|d^k - \nabla f(x^k)\|^2 \to 0\ as\ N \to \infty$ (12)
where $N \in (x_1, \dots, x_n)$ the data points.

4. CONVERGENCE OF THE STOCHASTIC APPROXIMATION ALGORITHM

Theorem 1: Let the sequence $\{a_k\}_{k\geq 0}$ of positive numbers satisfies the conditions:

(a) $a_0 = 1,\ 0 < a_k < 1 \quad \forall k > 1$
(b) $\sum_{k=0}^{\infty} a_k = \infty$
(c) $\sum_{k=0}^{\infty} (a_k)^2 < \infty$

Given that $\{d^k\}$ is a sequence of random vectors satisfying (9) and (10), and then the stochastic sequence $\{x^k\}_{k\geq 0}$ in H defined iteratively from $x_0 \in D(f)$ by $x^{k+1} = x^k - a_k d^k$ converges strongly to a unique solution $\{x^*: \nabla f(x^*) = 0\} \in H$ almost surely.

Proof: Let $\Lambda_k = a_k\|d^k - \nabla f(x^k)\|^2 = \frac{\sigma^2}{N}$, $0 < \sigma^2 < \infty$, then $\{\Lambda_k\}$ is a sequence of independent random variables. From (9), $E\Lambda_k = 0\ for\ each\ k,$ thus the sequence of partial sums $S_k = \sum_{i=1}^{k} \Lambda_i$ is a Martingale. But $E{S_k}^2 = \sum_{i=1}^{k} E{\Lambda_i}^2 = \sum_{i=1}^{k} a_i^2 E\left\|d^i - \nabla f(x^i)\right\|^2 = \frac{\sigma^2}{N}\sum_{i=1}^{k} a_i^2$. The convergence of the series $\sum a_i^2$ in (Theorem 1(c)),implies$\sum_{i=1}^{\infty} E\Lambda_i^2 < \infty$. Thus by Martingale convergence theorem (see [9]), we have $\sum_{k=1}^{\infty} \Lambda_k < \infty$, so that$\lim_{k\to\infty} a^k\|d^k - \nabla f(x^k)\| = 0 \implies d^k \to \nabla f(x^k)$.

It is obvious that the stochastic analog of (7) under suitable hypotheses, behavesAsymptotically as (7), and it implies the convergence properties of (7) are preservedwhen ∇f is replaced by d^k. It follows that the stochastic iteration scheme: $x^{k+1} = x^k - a^k d^k$ converges strongly to unique solution $\{x^*: \nabla f(x^*) = 0\} \in H$ From (10) d^k strongly approximates$\nabla f(x^k)\ at\ x^k$. Furthermore, it can be established that $\|d^k - \nabla f(x^k)\| \leq \left\|d_1^k - \nabla f(x^k)\right\|$ for all least-square approximations. Then the stochastic iterative scheme generated by the stochastic sequence $\{d^k\}$ is then given as follows:

Let x^k be an estimate of x^*

(a) Compute $d^k \approx \nabla f(x^k)$
(b) Compute a_k
(c) Compute $x^{k+1} = x^k - a_k d^k$
(d) Check for convergence:
Is $\|x^{k+1} - x^k\| < \lambda , \lambda > 0$?
Yes: $x^{k+1} = x^k$
No: Set $k = k + 1$ and return to (a)

5. Conclusion: Our aim in this paper is to prove a convergence result for approximating fixed points of a class of contractive-like operators defined in (4) using the iterative process (7). We prove a strong convergence result using stochastic iterative scheme with contractive operators associated with error terms. The iterative process is faster and simpler which improves other algorithms, and for the class of mapping in a more global sense.

6.Competing Interests: Author(s) have declared that no competing interests exist.

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